NONISOTHERMAL FLOW OF PLASMA IN A PLANE MHD-CHANNEL

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There is a large number of published papers (see the references given in the review [1]) in which various cases of the steady-state laminar flow of a conducting medium in a plane channel in the presence of a transverse magnetic field are considered. However, it has always been assumed that the transport coefficients were constants independent of the flow parameters such as the temperature. As a result, the dynamic and thermal problems were separable, and the temperature distribution had no effect on the dynamic flow parameters.

In a low-temperature dense plasma, the conductivity is a very rapidly increasing function of temperature (it is approximately given by $\sigma \sim e^{-A/T}$ or $\sigma \sim T^{10-13}$). It is clear that, in this case, it is necessary to take into account the fact that the transport coefficients are not constants, and the dynamic and thermal problems are not separable even for an incompressible fluid. We shall refer to such flows as nonisothermal, and contrast them with flows for which the transport coefficients are constants, and which we shall refer to as isothermal for the sake of brevity.

The importance of effects due to the nonisothermal nature of a flow was demonstrated in [2, 3]. Hysteresis effects in friction and heat transfer, which were found in these papers, were discussed qualitatively in [4]. Finally, the flow of a fluid with temperature-dependent conductivity in an MHD-channel was considered in [5] where it was noted that the nonisothermal nature of the flow must be taken into account. In particular, the appearance of nonmonotonic velocity profiles with points of inflection was demonstrated [5]. However, the latter paper included a number of conflicting assumptions. For example, when the propagation of heat was considered along the flow, it was assumed that the conductivity varied only across the channel. The temperature dependence of the conductivity used [5] was quite unrealistic, while the temperature dependence of viscosity and thermal conductivity was not taken into account at all.

In the present paper we investigate the flow of plasma in a plane MHD-channel in the absence of a longitudinal flow of heat but with allowance for the temperature dependence of the transport coefficients. We shall use a more realistic form of the temperature dependence for the above parameters, and will take viscous energy dissipation into account.

§1. We shall try to determine the velocity and temperature distributions u(y) and T(y) for steady flow of a viscous conducting fluid in a plane channel formed by parallel nonconducting walls $y = \pm a$. The walls are maintained at a constant temperature T_0. We shall suppose that the pressure gradient $\partial p/\partial x = -P < 0$ acts along the x-axis, while uniform magnetic and electric fields B and E are applied along the y and z axes. We shall further suppose that the applied fields do not violate the isotropy of the fluid parameters. If a steady-state solution which depends only on y exists, then it must satisfy the following equations which follow from the general system of magnetohydrodynamic equations:

$$\frac{d}{dy}\left(\eta\,\frac{du}{dy}\right)-\sigma(E+uB)B+P=0;$$
$$\frac{d}{dy}\left(\varkappa\,\frac{dT}{dy}\right)+\sigma(E+uB)^2+\eta\left(\frac{du}{dy}\right)^2=0. \qquad (1.1)$$

The corresponding boundary conditions are

$$u(\pm a)=0,\ T(\pm a)=T_0. \qquad (1.2)$$

When the transport coefficients σ, $\varkappa$, η are constants, then Eqs. (1.1) are linear, and the first of them can be solved independently of the second (this is the so-called Hartmann problem). In general, however, σ, $\varkappa$, η are functions of temperature, and the system of equations given by (1.1) is nonlinear, i.e., the dynamic problem cannot be separated from the thermal problem.

It is readily seen that three independent dimensionless combinations can be formed out of the parameters of the problem. We shall use the following quantities as the similarity criteria:

$$M=aB\sqrt{\frac{\sigma_0}{\eta_0}},$$
$$K=-\frac{E\eta_0}{BPa^2},\qquad N=\frac{P^2a^4}{\varkappa_0\eta_0T_0}. \qquad (1.3)$$

The velocity and temperature scales will be taken to be $u_P = Pa^2/\eta_0$ and $T_P = P^2a^4/\varkappa_0\eta_0$. Substituting

$$\xi=\frac{y}{a},\quad v=\frac{u}{u_P},\quad w=v-K,$$
$$\theta=\frac{T-T_0}{T_P},\ s=\frac{\sigma}{\sigma_0},\quad k=\frac{\varkappa}{\varkappa_0},\quad h=\frac{\eta}{\eta_0}.$$

where σ_0, $\varkappa_0$, η_0 are the values of σ, $\varkappa$, η for $T = T_0$, and denoting differentiation with respect to ξ by a prime, we can rewrite Eqs. (1.1) in the dimensionless form

$$(hw')'-M^2sw+1=0,$$
$$(k\theta')'+M^2sw^2+h(w')^2=0. \qquad (1.4)$$

Since the problem is symmetric with respect to ξ, it is sufficient to find the solution in the interval (0, 1). The boundary conditions must then be taken in the form

$$w'(0)=0,\ w(1)=-K,\ \theta'(0)=0,\ \theta(1)=0. \qquad (1.5)$$

Only the parameters σ, $\varkappa$, η will depend on the absolute temperature, and therefore N will appear only in the functions s, k, h. In the linear problem, these functions are identically equal to unity, and the quantity N cancels out. Here we shall assume that

$$\sigma=\sigma_0\left(\frac{T}{T_0}\right)^{\alpha},\qquad \varkappa=\varkappa_0\left(\frac{T}{T_0}\right)^{\beta},\qquad \eta=\eta_0\left(\frac{T}{T_0}\right)^{\gamma}. \qquad (1.6)$$

or

$$s=(1+N\theta)^{\alpha},\quad k=(1+N\theta)^{\beta},\quad h=(1+N\theta)^{\gamma}. \qquad (1.7)$$

This case is of particular interest for plasma flow analysis. Thus, for fully ionized plasma $\alpha = 3/2$, $\beta = \gamma = 5/2$, while for weakly ionized gas the electrical conductivity is a rapidly varying function of temperature, which can be approximated by a power function with $\alpha \sim 10$. When $\alpha = \beta = \gamma = 0$, the problem degenerates to the linear problem for which we have two similarity criteria, namely, M and K. The exponents α, β, γ will be regarded as additional similarity criteria. Therefore, the solution of the nonlinear boundary-value problem defined by Eqs. (1.4) and (1.5) depends on six dimensionless parameters, i.e., it is of the form

$$v = v(\xi;\ K,\ M,\ N,\ \alpha,\ \beta,\ \gamma),$$
$$\theta = \theta(\xi;\ K,\ M,\ N,\ \alpha,\ \beta,\ \gamma). \quad (1.8)$$

In the general case, the solution can be obtained only by numerical methods. For relatively small values of M and N it is possible to use the methods of conjugate equations with iteration [6, 7]. For large M and N, the method of finite differences has been found to be the only acceptable one.

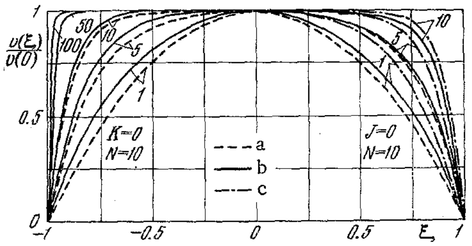

Fig. 1

The problem can also be formulated in another way by specifying the current I per unit length of the channel, instead of the field E. It is then convenient to replace K by the dimensionless mean current density in the channel

$$J = IB / 2Pa, \quad (1.9)$$

so that the boundary conditions (1.5) are replaced by

$$w'(0) = 0, \quad w'(1) = J - 1,$$
$$\theta'(0) = 0, \quad \theta(1) = 0. \quad (1.10)$$

The current is therefore uniquely related to the derivative of the velocity at the wall. From Eqs. (1.4) we can obtain an expression for the mean dimensionless velocity (one half of the flow rate of the fluid)

$$V = \frac{1}{a}\int_0^a \frac{u(y)}{u_P}\,dy =$$
$$= w(1)\,[1 + w'(1)] - \theta'(1) = KJ + Q, \quad (1.11)$$

where Q is the dimensionless heat flow to the channel wall.

The following is a brief classification of the possible flow states:

(1) $V < 0,\ K < 0,\ J > 0$ — conductive pumping;
(2) $V = 0,\ K < 0,\ J > 0$ — channel cutoff;
(3) $V > 0,\ K < 0,\ J > 0$ — electromagnetic drag;
(4) $V > 0,\ K = 0,\ J > 0$ — short circuited MHD-generator;
(5) $V > 0,\ K > 0,\ J > 0$ — MHD-generator mode;
(6) $V > 0,\ K > 0,\ J = 0$ — zero-load MHD-generator
(7) $V > 0,\ K > 0,\ J > 0$ — MHD-accelerator

Henceforth we shall pay particular attention to cases (4) and (6).

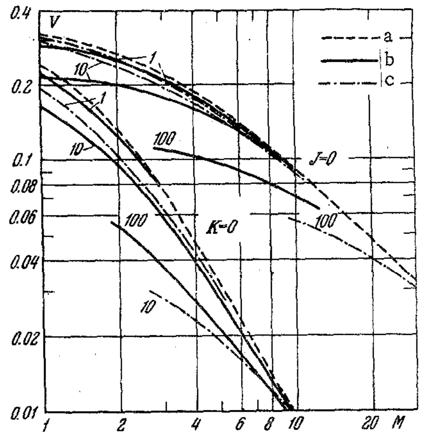

Fig. 2

§2. For the heat transfer problem which we are considering, the temperature distribution is monotonic and the plasma transport coefficients increase monotonically toward the center of the channel. This means that, other things being equal, the current density and the ponderomotive force at the center of the flow should exceed the corresponding quantities in the isothermal case, i.e., the fact that the flow is nonisothermal leads to a flattening of the velocity profile and to a reduction of its mean value if the ponderomotive force retards the flow. This effect increases with increasing N and α.

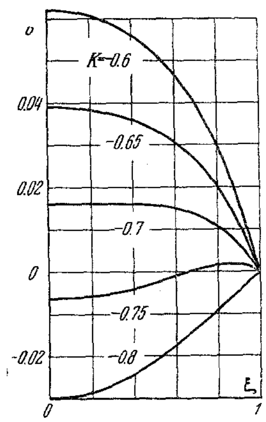

Fig. 3

The degree of deformation of the velocity profile and the change in its magnitude are also found to depend on the flow conditions. For example, when K = 0, the current density increases toward the center without changing sign across the channel. An increase in the conductivity elongates the current profile still further, continuously increasing the retarding force at the center. For J = 0, when the current density does change sign across the flow, there may be an additional nonisothermal retardation of the flow core, while acceleration by the ponderomotive force in the outer regions increases more slowly. Although in both cases the result is a more extended velocity profile, it is expected that nonisothermal effects are more clearly defined when K = 0. The above discussion is illustrated by the velocity profiles shown in Fig. 1. The numbers shown against the various curves represent the values of M.

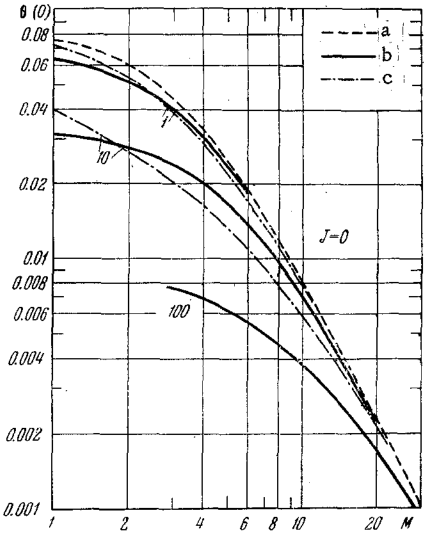

Fig. 4

The following correspondence between the values of a, β, γ and the shape of the curves is used: (a) $a = \beta = \gamma = 0$, (b) $a = {}^3/_2$, $\beta = \gamma = {}^5/_2$, (c) $a = 10$, $\beta = \gamma = 0$. When J = 0 and $M \geq 10$, the solid and dashed lines coincide, while for K = O and M = 10 there is still an appreciable difference between them. Figure 2 shows the variation of the mean velocity V with M for the same cases (the numbers shown against the curves are the values of N). It is clear that, when the nonisothermal effects are taken into account, there is a reduction in the flow rate, and the difference increases with increasing N and α. As M increases, there is a stronger deceleration of the flow for K = 0 than for J = 0, as in the isothermal case.

The difference between the linear and nonlinear problems can be seen most clearly under the channel cutoff conditions. Thus, it is well known that in the isothermal case with $K = -M^{-2}$, both V and v change sign simultaneously for all values of ξ. In the nonisothermal case, this is impossible, and a transition region appears in the neighborhood of V = 0 in which $v(\xi)$ changes sign, i.e., the velocity profile has points of inflection. Figure 3 shows the effect of a change in K on the velocity profile for M = 1, N = 1, $a = {}^3/_2$, $\beta = \gamma = {}^5/_2$.

We note that nonmonotonic velocity profiles with points of inflection are also found to appear as N increases under other flow conditions, for example, for K = 0, N $\gtrsim$ 100. Their shape can be explained by the distribution of temperature, currents, and forces in each specific case, although it is not clear whether such states can be realized in practice. If the conditions necessary for the validity of the Rayleigh-Tollmien theorem are satisfied, the theorem definitely indicates that these states are unstable. This will evidently take place when the parameter $S = M^2/R$ and the magnetic Reynolds number $R_m = \sigma_0 \mu u a$ are small. In other cases, the problem requires special analysis.

Figure 4 shows the maximum temperature $\theta(0)$ as a function of M for J = 0. (When K = 0 there is an analogous situation.) We note that when the nonisothermal effects are taken into account, there is a reduction in the temperature in the channel for both fully ionized and weakly ionized plasma. The physical reason for this is the reduction in Joule heating due to the increase in conductivity. The reduced heating is very dependent on N (see numbers on the curves).

§3. Finally, consider the effect of nonisothermal conditions on the coefficients of skin friction and heat transfer to the wall. The problem does not depend on the density of the fluid (and the solutions are formally valid for any R). Therefore, it is convenient to replace the usual friction coefficient

$$C_f = \frac{-\eta_0 [du/dy]_{y=a}}{\rho u^2(0)/2}$$

by the dimensionless combination

$$C_f R / 2 = -v'(1) / v(0). \qquad (3.1)$$

The dimensionless heat transfer coefficient is given by

$$C_q = -\frac{\varkappa_0}{q^\circ}\left[\frac{dT}{dy}\right]_{y=a} = -\frac{\theta'(1)}{\theta(0)}$$

$$\left(q^\circ = \varkappa_0 \frac{T(0) - T_0}{a}\right),$$

where the heat-flow scale is chosen to be q°.

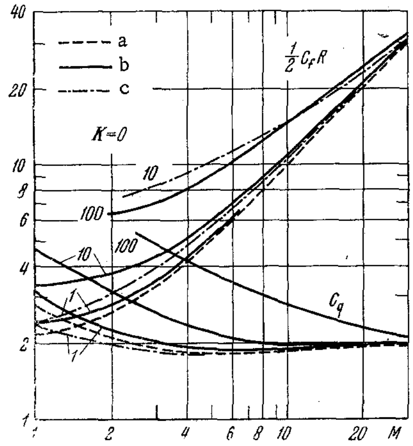

Fig. 5

The quantities given by Eqs. (3.1) and (3.2) are shown in Fig. 5 as functions of M for K = 0 (the numbers shown against the curves represent the values of N). As expected, since the velocity profiles are extended, the frictional resistance in the nonisothermal case is greater, and the difference increases with increasing N and α. The descending curves show the behavior of C_q. When J = 0, the behavior of the curves for the friction coefficient remains the same, while the behavior of C_q changes radically: the $C_q(M)$ curves are similar to the skin friction terms. All this is readily understood by considering v and θ for large M when K = 0, J = 0 in the isothermal problem, where all the formulas are elementary.

We note in conclusion that the above nonisothermal effects are in many respects due to the assumed positive values of α, β, and γ, which is valid for plasmas. For other media, for example, liquid metals, α can be negative, and it will not be surprising if some of the above effects change their magnitude and even sign.

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